Journal of Engineering Physics and Thermophysics, Vol. 82, No. 3, 2009

SOLUTION OF PHASE-TRANSFORMATION PROBLEMS BY THE METHOD OF EXTENSION OF BOUNDARIES

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UDC 536.539.517.946

The efficiency of the analytical method of extension of boundaries is shown using phase transformations as an example. The initial problem is replaced by an auxiliary one where rectangular regions with known eigenfunctions and eigenvalues are made to correspond to curvilinear regions with moving boundaries of each phase. This enables us to represent the solution by expansions in improved Fourier series differing from the classical ones by an increased convergence rate. Finally, the problem is reduced to a small number of differential equations of first order in time.

Keywords: *method of extension of boundaries, phase transformations, improved Fourier series.*

Introduction. Versions of moving-boundary problems are problems on phase transformations [1]. Such problems are difficult to mathematically analyze, since not only the temperatures in both phases are unknown (i.e., two unknowns now, not one) but the curvilinear moving boundary Γ^* separating the phase regions Ω_1 and Ω_2, at which the boundary conditions due to the phase transition must be fulfilled, is also unknown. The most substantial results in this field have been obtained in G. A. Grinberg and É. M. Kartashov's works, reviewed in sufficient detail in [2]. Several exact solutions of multidimensional single-phase Stefan problems have been obtained in [3].

Formulation of the Problem and Its Solution. For simplicity we consider a plane problem in variables (x, y), although the method can be applied analogously to spatial problems with phase transformations. The case where the entire material is in the state of just the 1st phase and the 2nd phase begins to be initiated under certain condition will not be considered here. Consequently, from the very beginning of the process of phase transformation, both phases of the given material are in the simply connected region Ω.

We prescribe the law of motion of the boundary Γ of the region Ω in parametric form

$$\Gamma \Rightarrow x = x_\Gamma(t, \theta)\,, \quad y = y_\Gamma(t, \theta)\,, \quad 0 \le \theta \le 2\theta_0\,,$$

$$\left[x_\Gamma(t, \theta), y_\Gamma(t, \theta)\right] \in C^{(1)}\,(0 \le t \le t_0, 0 \le \theta \le 2\theta_0)\,. \tag{1}$$

We will assume that as the parameter θ varies within $[0, \theta_0]$, the point (x_Γ, y_Γ) traces the entire boundary Γ, and for $\theta \in [\theta_0, 2\theta_0]$, it traces it for the second time. The boundaries Γ_1 and Γ_2 can be both moving and stationary, which is determined by prescribing dependences (1). The entire region Ω is separated here into two simply connected parts $\Omega_1(t)$ and $\Omega_2(t)$ corresponding to the 1st and 2nd phases (see Fig. 1). The boundary Γ of the region Ω consists of the parts Γ_1 and Γ_2. The regions Ω_1 and Ω_2 are separated by an unknown moving phase-transformation boundary $\Gamma^*(t)$, which intersects Γ at two points A_u^* (x_{au}^*, y_{au}^*) (u = 1 and 2) whose coordinates are unknown, too. To find them we write the equation of the Γ^* curve for an arbitrary t and its initial position at t = 0:

$$\Gamma^*(t) \Rightarrow y = f^*(t, x)\,, \quad \Gamma^*\big|_{t=0} \Rightarrow y = f_0^*(x)\,, \quad [f^*, f_0^*] \in C^{(1)}\,(0 \le t \le t_0, x_{a1}^* \le x \le x_{a2}^*)\,, \tag{2}$$

where $f^*(t, x)$ is the unknown function and $f_0^*(t, x)$ is the prescribed function as the initial position of the phase boundary Γ^* at t = 0. The boundaries Γ_1 and Γ_2 unite at two points A_u^* whose coordinates can be determined from the condition of intersection of the Γ and Γ^* curves. Substituting x_Γ and y_Γ from (1) into (2), we obtain

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1062-0125/09/8203-0574

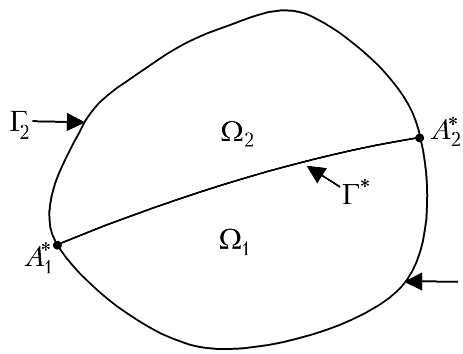

Fig. 1. Curvilinear region Ω consisting of the phase regions Ω_1 and Ω_2 that are separated by the phase-transition boundary Γ^*.

$$y_\Gamma(t, \theta_u^*) = f^*\left(t, x_\Gamma(t, \theta_u^*)\right), \quad 0 \le \theta_u^* \le \theta_0, \; u = 1, 2. \tag{3}$$

Hence we find two values of the parameter $\theta_u^* = \theta_u^*(t)$, $u = 1$ and 2, as functions of the time t, which determine the coordinates of two moving points A_u^*:

$$x_{au}^* = x_\Gamma\left(t, \theta_u^*(t)\right), \quad y_{au}^* = y_\Gamma\left(t, \theta_u^*(t)\right), \quad u = 1, 2. \tag{4}$$

We will assume that the temperature $T_1(t, x)$ in the region Ω_1 of the first phase is lower, and $T_2(t, x)$ in the region Ω_2 of the second phase is higher than the phase-transformation temperature T^*; these temperatures are coincident at their common boundary Γ^* and are equal to T^*. Furthermore, the energy balance between the heat fluxes and the heat of phase transition must hold, i.e., we must have

$$T_1\big|_{\Gamma^*} = T_2\big|_{\Gamma^*} = T^* = \text{const}, \quad \left[\lambda_1 \frac{\partial T_1}{\partial n^*} - \lambda_2 \frac{\partial T_2}{\partial n^*}\right]\Bigg|_{\Gamma^*} = h^* v_n,$$
$$T_1\left(\forall (x, y) \in \Omega_1\right) \le T^* \le T_2\left(\forall (x, y) \in \Omega_2\right). \tag{5}$$

The normal projection of the velocity v_n can be expressed by the equation of the moving boundary Γ^*:

$$v_n = \frac{\partial f^*(t, x)}{\partial t} \Big/ N_0, \quad N_0 = \sqrt{1 + \left(\frac{\partial f^*(t, x)}{\partial x}\right)^2}. \tag{6}$$

If we take the vector $\left(-\frac{1}{N_0}\frac{\partial f^*(t, x)}{\partial x}, \frac{1}{N_0}\right)$ directed toward the second phase as the normal n^* (see Fig. 1), the phase-transformation conditions in (5) will take a more convenient form:

$$T_1\big|_{\Gamma^*} = T_2\big|_{\Gamma^*} = T^*, \quad T_1\left(\forall (x, y) \in \Omega_1\right) \le T^* \le T_2\left(\forall (x, y) \in \Omega_2\right),$$
$$\left[\lambda_1\left(\frac{\partial T_1}{\partial y} - \frac{\partial T_1}{\partial x}\frac{\partial f^*(t, x)}{\partial x}\right) - \lambda_2\left(\frac{\partial T_2}{\partial y} - \frac{\partial T_2}{\partial x}\frac{\partial f^*(t, x)}{\partial x}\right)\right]\Bigg|_{\Gamma^*} = h^* \frac{\partial f^*(t, x)}{\partial t}. \tag{7}$$

Here the sign of the expression in square brackets determines the direction of the process: if it is positive, we have $v_n > 0$ and then the second phase changes to the first phase, whereas for a negative sign of $v_n < 0$, the first phase becomes the second phase. For simplicity the material particles will be considered not to move in each phase and the heat-conduction equations can be written in the form

$$D_i\,(T_i) = q_i\,(t, x, y)\,, \quad (x, y) \in \Omega_i\,, \quad t \in [0, t_0]\,; \qquad (8)$$

$$T_i \in L_p^{\alpha+2}\,(\Omega_i)\,, \quad T_i \in C^{(2)}\,(\Omega_i)\,, \quad i = 1, 2\,.$$

The differential operators D_i in (8) can be different if the material has unequal physical properties in different phase states. In addition to (7), we should write, for Eqs. (8), the initial and boundary conditions at Γ_1 and Γ_2:

$$L_i\,(T_i)\big|_{\Gamma i} = f_{\Gamma i}\,(t, x_\Gamma, y_\Gamma)\,, \quad (x_\Gamma, y_\Gamma) \in \Gamma_i\,, \quad T_i\big|_{t=0} = f_0^{(i)}\,(x, y)\,, \qquad (9)$$

$$\left(f_0^{(i)}, f_{\Gamma i}\right) \in C^{(2)}\,(\Omega_i)\,, \quad \left(f_0^{(i)}, f_{\Gamma i}\right) \in L_p^{\alpha}\,(\Omega_i)\,, \quad i = 1, 2\,.$$

The smoothness and integrability conditions in (8) and (9) are dictated by the necessity of differentiating and the possibility of expanding the corresponding functions in generalized Fourier series [4] in the process of obtaining the solution. For the formulation of the problem to the completed, it is necessary that at the two points A_u^* (x_{au}^*, y_{au}^*), u = 1 and 2 (see Fig. 1), where three boundaries Γ_1, Γ_2, and Γ^* intersect, the boundary conditions at them be matched, i.e.,

$$\left[L_i\,(T_i) = f_{\Gamma i}\,(t, x, y)\right]\big|_{A_j^*} \sim \left\{ \begin{array}{c} \left[\lambda_1 \left(\dfrac{\partial T_1}{\partial y} - \dfrac{\partial T_1}{\partial x}\dfrac{\partial f^*}{\partial x}\right) - \lambda_2 \left(\dfrac{\partial T_2}{\partial y} - \dfrac{\partial T_2}{\partial x}\dfrac{\partial f^*}{\partial x}\right)\right] = h^* \dfrac{\partial f^*}{\partial t} \\ T_1 = T_2 = T^* \end{array} \right\}\Bigg|_{A_j^*}\,, \qquad (10)$$

where ~ means "matching." Thus, e.g., in the case where the Dirichlet conditions are specified at Γ_1 and Γ_2, the functions $f_{\Gamma i}$ (t, x, y) at the points A_u^* (x_{au}^*, y_{au}^*), u = 1 and 2, in accordance with conditions (10), must satisfy the equalities

$$f_{\Gamma i}\left(t, x_{au}^*, y_{au}^*\right) = T^*\,, \quad i, u = 1, 2\,. \qquad (11)$$

In problem (1)–(10), the unknown functions are the temperatures T_i in the phase regions Ω_i and the equation of the phase boundary $f^*(t,\ x)$.

Had the boundary Γ^* been known, the first two boundary conditions of the three in (7) in combination with boundary and initial conditions (9) would have closed the problem on finding T_i (i = 1 and 2) by solution of the differential equations (8). To find $f^*(t,\ x)$ we use the supplementary third equation from (7), i.e., the phase-transformation condition.

To solve the entire problem in accordance with the method of extension of boundaries we introduce two auxiliary classical simply connected regions $\Omega_{\square 1}$ and $\Omega_{\square 2}$ which are wider than $\Omega_1(t)$ and $\Omega_2(t)$, i.e.,

$$\Omega_1\,(t) \in \Omega_{\square 1}\,, \quad \Omega_2\,(t) \in \Omega_{\square 2}\,, \quad t \in [0, t_0]\,,$$

here, $\Omega_{\square 1}$ and $\Omega_{\square 2}$ can intersect. Their closed boundaries will be denoted by $\Gamma_{\square 1}$ and $\Gamma_{\square 2}$. We select, as $\Omega_{\square i}$, two rectangles of the same width a along the x axis and of height b_i (i = 1 and 2) along the y axis

$$\Omega_{\square i} = (0 \le x \le a) \times (0 \le y - y_{0i} \le b_i)\,, \quad y_{01} = 0\,, \qquad (12)$$

where the dimensions y_{02}, a, and b_i (i = 1 and 2) should be selected so that the regions Ω_i do not go beyond their rectangular regions $\Omega_{\square i}$ (0, y_{0i}) over the period t_0 and (0, y_{0i}) are the coordinates of the lower left angle of the corresponding rectangle. From further presentation of the method, it will be evident that when the dependence for the phase-transition boundary in the form (2) is used, expansion of the function $f^*(t,\ x)$ in a Fourier series is substantially simplified if both rectangles $\Omega_{\square 1}$ and $\Omega_{\square 2}$ of the same width along the x axis are selected so that $\Omega_{\square 1}$ and $\Omega_{\square 2}$ are located in the same band $0 \le x \le a$. In the case where the equation of the boundary $\Gamma^*(t)$ is written, instead of (2), by the equality $x = g^*(t, y)$ we should select the same height of both rectangles along the y axis.

We replace the problem on phase transformation (1)–(10) for the region Ω by an auxiliary phase-transformation problem in the region $\Omega_{\square 1} \cup \Omega_{\square 2}$ with boundary conditions at $\Gamma_{\square 1}$ and $\Gamma_{\square 2}$, which are not known in advance. Phase transition is assumed at a certain boundary $\Gamma^*_{\square}(t)$, which will be determined so that both $\Gamma^*(t)$ and $\Gamma^*_{\square}(t)$ are coincident at common points for the points $(x, y) \in \Omega$. For this purpose we continue (extend) the phase boundary $\Gamma^* \in \Omega$ into the extended part of the region $\Omega_{\square i} \setminus \Omega$ ($i = 1$ and 2):

$$y = f^*_{\square}(t, x)\,, \quad f^*_{\square}(t, x)\big|_{t=0} = f^*_{\square 0}(x)\,, \quad f^*_{\square}(t, x) \in C^{(2)}\,(0 \le t \le t_0,\, 0 \le x \le a)\,, \tag{13}$$

where $f^*_{\square}(t, x)$ is the unknown function and $y = f^*_{\square 0}(x)$ is the equation of the initial position of $\Gamma^*_{\square}(0)$, which is prescribed.

We make the auxiliary problem with Dirichlet boundary conditions (for definiteness, although we can use other linear boundary conditions) correspond to problem (1)–(10):

$$\begin{gathered} D_i\,(T_{\square i}) = q_{\square i}\,(t, x, y)\,, \quad T_{\square i}\big|_{\Gamma_{\square i}} = f_{\square \Gamma i}\,(t, x_{\Gamma \square i}, y_{\Gamma \square i})\,, \quad T_{\square i}\big|_{t=0} = f^{(i)}_{\square 0}\,(x, y)\,, \\ (x, y) \in \Omega_{\square i}\,, \quad t \in [0, t_0]\,, \quad T_{\square i} \in L^{\alpha+2}_p\,(\Omega_{\square i})\,, \quad T_{\square i} \in C^{(2)}\,(\overline{\Omega}_{\square i})\,, \quad i = 1, 2\,. \end{gathered} \tag{14}$$

Now, instead of (7), the phase-transformation conditions should be fulfilled at $\Gamma^*_{\square}$:

$$\begin{gathered} T_{\square 1}\Big|_{\Gamma_{\square}^*} = T_{\square 2}\Big|_{\Gamma_{\square}^*} = T^*\,, \quad \Gamma^*_{\square} \Rightarrow y = f^*_{\square}\,(t, x)\,, \\ \left[\lambda_1\left(\frac{\partial T_1}{\partial y} - \frac{\partial T_1}{\partial x}\frac{\partial f^*_{\square}}{\partial x}\right) - \lambda_2\left(\frac{\partial T_2}{\partial y} - \frac{\partial T_2}{\partial x}\frac{\partial f^*_{\square}}{\partial x}\right)\right]\Bigg|_{\Gamma_{\square}^*} = h^*\,\frac{\partial f^*_{\square}}{\partial t}\Bigg|_{\Gamma_{\square}^*}\,. \end{gathered} \tag{15}$$

Similarly to (8), we impose additional conditions on the operators D_i in the extended regions $\Omega_{\square i}$:

$$\text{if} \quad F\,(x, y) \in L^{\alpha+2}_p\,(\Omega_{\square i})\,, \quad \text{then} \quad D_i\,\big(F\,(x, y)\big) \in L^{\alpha}_p\,(\Omega_{\square i})\,. \tag{16}$$

We should continue the right-hand sides $q_{\square i}$ of the differential equations (8), the initial conditions $f^i_{\square 0}$, and the function $f^*_0(x)$ determining the initial position of the boundary Γ^* into the region $\Omega_{\square i} \setminus \Omega$ ($i = 1$ and 2), supplementing the definition with the equalities

$$\begin{gathered} \left(q_{\square i}, f^{(i)}_{\square 0}, f^*_{\square 0}\right) = \begin{cases} \left(q_i, f^{(i)}_0, f^*_0\right), & \text{if} \quad (x, y) \in (\Omega_i)\,, \\ \left(\tilde{q}_i, \tilde{f}^{(i)}_0, \tilde{f}^*_0\right), & \text{if} \quad (x, y) \in (\Omega_{\square i} \setminus \Omega)\,, \end{cases} \\ \left(q_{\square i}, f^{(i)}_{\square 0}, f^*_{\square 0}\right) \in L^{\alpha}_p\,(\Omega_{\square i})\,, \quad \left(q_{\square i}, f^{(i)}_{\square 0}, f^*_{\square 0}\right) \in C^{(1)}\,(\Omega_{\square i})\,, \quad f^*_{\square 0} \in C^{(2)}\,(\Omega_{\square i})\,. \end{gathered} \tag{17}$$

The variables $\tilde{q}_i$, $\tilde{f}^{(0)}_0$, and $\tilde{f}^*_{\square 0}$ are prescribed by selection so that the smoothness and integrability conditions indicated in (17) are fulfilled for $q_{\square i}$, $f^{(i)}_{\square 0}$, and $f^*_{\square 0}$. In the cases where these functions are prescribed in analytical form we can use analytical continuations into $\Omega_{\square i} \setminus \Omega$ ($i = 1$ and 2) for them.

The boundary conditions $f_{\square \Gamma i}(t, x_{\Gamma \square}, y_{\Gamma \square})$ at the boundaries $\Gamma_{\square i}$ from (14) in the auxiliary problem are assumed to be not known in advance and will be determined so that the problem's solution at the corresponding moving boundaries $\Gamma_i(t)$ takes on the values of the boundary conditions of the initial problem (1)–(10), i.e.,

$$L_i\,(T_{\square i})\Big|_{\Gamma i} = f_{\Gamma i}\,, \quad (x_\Gamma, y_\Gamma) \in \Gamma_i\,, \qquad T_{\square i}\Big|_{\Gamma \square i} = f_{\square \Gamma i}\,, \quad (x_{\Gamma \square}, y_{\Gamma \square}) \in \Gamma_{\square i}\,, \quad i = 1, 2\,. \tag{18}$$

Thus, the method of extension of boundaries involves the formulation of the following problem: *to find boundary conditions $f_{\square\Gamma i}$ at the boundaries of the rectangles $\Gamma_{\square i}$ of the extended regions $\Omega_{\square i}$ and a solution $T_{\square i}$ of the auxiliary problem (13)–(18) such that they satisfy all the conditions of the initial problem (1)–(10).*

To find $T_{\square i}$ we select the most convenient extended regions $\Omega_{\square i}$ and construct the solution of problem (13)–(18) with arbitrary boundary conditions $f_{\square\Gamma i}$ not known in advance. In formulating problem (13)–(18), we have $T_{\square i} \in C^{(2)}(\overline{\Omega}_{\square i})$; therefore, computation of the nonlinear operators $L_i\ (T_{\square i})\big|_{\Gamma i}$ is legitimate since the points of the boundaries Γ_i belong to the corresponding extended regions $\Omega_{\square i}$ where the assumed Fourier series for $T_{\square i}$ uniformly converge. Problem (13)–(18) is more simple than (1)–(10), since either stationary regions $\Omega_{\square i}$ are selected or their boundaries move by preprescribed laws.

The ranges of eigenfunctions and eigenvalues for the rectangular regions $\Omega_{\square i}$ will have the form

$$G_{m,n}^{(i)} = \sin m\pi \frac{x}{a} \sin n\pi \frac{y - y_{0i}}{b_i}\,, \quad (0 \le x \le a) \times (0 \le y - y_{0i} \le b_i)\,,$$

$$\mu_{m,n}^{(i)} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b_i}\right)^2, \quad (m, n) = 1, 2, \dots\,, \quad i = 1, 2\,. \tag{19}$$

We perform the replacement

$$T_{\square i}\,(t, x) = M_i\,(t, x) + V_i\,(t, x)\,, \quad (M_i, V_i) \in L_p^{\alpha+2}\,(\Omega_{\square i})\,, \quad t \in [0, t_0]\,, \quad i = 1, 2\,, \tag{20}$$

where V_i are the new unknowns and M_i are boundary functions selected so as to satisfy the boundary conditions for $T_{\square i}$ in the formulation of the auxiliary problem (13)–(18). For this purpose we write the Dirichlet conditions on the sides of the rectangles in general form

$$T_{\square i}\big|_{\Gamma\square} = M_i\big|_{\Gamma\square} \Rightarrow M_i\big|_{y=y_{0i}} = f_1^{(i)}\,(t, x)\,, \quad M_i\big|_{x=a} = f_2^{(i)}\,(t, y)\,, \quad i = 1, 2\,, \quad t \in [0, t_0]\,;$$
$$M_i\big|_{y=y_{0i}+b_i} = f_3^{(i)}\,(t, x)\,, \quad M_i\big|_{x=0} = f_4^{(i)}\,(t, y)\,, \quad f_j^{(i)} \in L_p^{\alpha+2}\,(\Omega_{\square i})\,, \quad j = 1\text{–}4\,, \tag{21}$$

where $f_j^{(i)}$ are eight functions (not known in advance) of the time t and of one geometric coordinate corresponding to their domain of definition. The continuity of M_i at the angles of the rectangles, as they are approached from both directions along the sides of a given angle, yields eight conditions of matching for the functions $f_j^{(i)}$:

$$f_1^{(i)}\,(t, 0) = f_4^{(i)}\,(t, y_{0i})\,, \quad f_1^{(i)}\,(t, a) = f_2^{(i)}\,(t, y_{0i})\,, \quad t \in [0, t_0]\,;$$
$$f_2^{(i)}\,(t, y_{0i} + b_i) = f_3^{(i)}\,(t, a)\,, \quad f_3^{(i)}\,(t, 0) = f_4^{(i)}\,(t, y_{0i} + b_i)\,, \quad i = 1, 2\,. \tag{22}$$

In addition to (22), the functions $f_j^{(i)}$ must satisfy supplementary conditions identical to the matching relations (10) for the basic problem. Therefore, from the auxiliary problem, boundary conditions (21) at $\Gamma_{\square i}$ and (15) at $\Gamma_{\square i}^*$ at the points $B_{\square u}^*$ must also be matched. To obtain these conditions we write the coordinates of two points $B_{\square 1}^*$ and $B_{\square 2}^*$ of intersection of the phase boundary $\Gamma_{\square}^*$ and the rectangle sides $x = 0$ and $x = $ a;

$$B_{\square 1}^* \to x_{b1}^* = 0\,, \quad y_{b1}^* = f_{\square}^*\,(t, 0)\,; \quad B_{\square 2}^* \to x_{b2}^* = a\,, \quad y_{b2}^* = f_{\square}^*\,(t, a)\,. \tag{23}$$

Then the matchings of boundary conditions (21) and conditions (15) for the auxiliary problem take the form of four additional equations

$$f_4^{(i)}\,(t, y_{b1}^*) = f_2^{(i)}\,(t, y_{b2}^*) = T^*\,, \quad i = 1, 2\,, \tag{24}$$

where y_{b1}^*, y_{b2}^* should be taken from (23). Had the boundary $\Gamma_\square^*$ been located in a band parallel to the x axis, we would have had, instead of (24), four similar equalities for the functions $f_1^{(i)}$ and $f_3^{(i)}$.

In accordance with boundary conditions (21), the functions M_i will be represented by the dependences

$$M_i = \left(1 - \frac{y - y_{0i}}{b_i}\right)\left[f_1^{(i)}(t, x) - \frac{x}{a} f_1^{(i)}(t, a) - \left(1 - \frac{x}{a}\right) f_1^{(i)}(t, 0)\right] + \frac{x}{a} f_2^{(i)}(t, y)$$
$$+ \frac{y - y_{0i}}{b_i}\left[f_3^{(i)}(t, x) - \frac{x}{a} f_2^{(i)}(t, b_i) - \left(1 - \frac{x}{a}\right) f_3^{(i)}(t, 0)\right] + \left(1 - \frac{x}{a}\right) f_4^{(i)}(t, y) . \tag{25}$$

Thus, in constructing the solution of the phase-transformation problem, we must find the following 11 functions:

$$\left\{V_i(t, x, y), f_1^{(i)}(t, x), f_2^{(i)}(t, y), f_3^{(i)}(t, x), f_4^{(i)}(t, y)\right\} \in \left\{C^{(2)}(\Omega_i), L_p^\alpha(\Omega_i), 0 \le t \le t_0\right\} ,$$

$$f_\square^*(t, x) \in \left\{C^{(1)}(\Omega_i), L_p^\alpha(\Omega_i), 0 \le t \le t_0\right\} , \quad i = 1, 2 .$$

We represent them as uniformly convergent Fourier series in the corresponding regions, where we confine ourselves to a finite number of terms:

$$V_i = \sum_{m,n=1}^{m_0, n_i} V_{m,n}^{(i)}(t)\, G_{m,n}^{(i)} , \quad (x, y) \in \Omega_{\square i} ; \quad f_j^{(i)} = \frac{x}{a}\left[f_j^{(i)}(t, a) - f_j^{(i)}(t, 0)\right]$$

$$+ f_j^{(i)}(t, 0) + \sum_{m=1}^{m_0} f_{j,m}^{(i)}(t) \sin m\pi \frac{x}{a} , \quad x \in [0, a] , \quad j = 1, 3 , \quad i = 1, 2 ; \quad f_{j+1}^{(i)} = f_{j+1}^{(i)}(t, y_{0i}) \tag{26}$$

$$+ \frac{y - y_{0i}}{b_i}\left[f_{j+1}^{(i)}(t, y_{0i} + b_i) - f_{j+1}^{(i)}(t, y_{0i})\right] + \sum_{n=1}^{n_i} f_{j+1,n}^{(i)}(t) \sin n\pi \frac{y - y_{0i}}{b_i} ;$$

$$y - y_{0i} \in [0, b_i] , \quad f_\square^* = f_\square^*(t, 0) + \frac{x}{a}\left[f_\square^*(t, a) - f_\square^*(t, 0)\right] + \sum_{s=1}^{s^*} f_{\square s}^*(t) \sin s\pi \frac{x}{a} .$$

The coefficients of Fourier expansions in (26) are determined from standard formulas. Here all functions have been determined either for $0 \le x \le a$, or for $0 \le y - y_{0i} \le b$, or for $(x, y) \in \Omega_i$; therefore, their presented expansions in Fourier series in sines are legitimate. The expansions (26) are so constructed that these series uniformly converge not only inside the corresponding regions but at their boundaries as well. It is well known [5] that a Fourier series in sines for smooth functions can singly be differentiated termwise if the series converges at the boundaries, which is the case. After differentiation, we obtain cosine series which, according to the corresponding theorem [5], allow termwise differentiation, too. Thus, the series (26) can be doubly differentiated with respect to geometric coordinates. The functions in (26) are dependent on the time t as on the parameter; therefore when the conditions of their smoothness are fulfilled, the series (26) allow termwise differentiation with respect to t as with respect to the parameter. We note that the convergence of the series (26) at the boundaries contribute to their rapid general convergence, which substantially reduces computation time and improves the exactness of the approximate solution.

Using the expansions (26) we reduce the solution of the auxiliary problem (13)–(21) to finding the following coefficients and two parameters θ_u^* (u = 1 and 2) dependent just on time:

$$V_{m,n}^{(i)}(t)\,,\;\; f_{j,m}^{(i)}(t)\,,\;\; f_{j+1,n}^{(i)}(t)\,,\;\; f_{\square s}^{*}(t)\,,\;\; f_{j}^{(i)}(t,0)\,,\;\; f_{j}^{(i)}(t,a)\,,\;\; f_{j+1}^{(i)}(t,y_{0i})\,,\;\; f_{j+1}^{(i)}(t,y_{0i}+b_i)\,,$$

$$f_{\square}^{*}(t,0)\,,\;\; f_{\square}^{*}(t,a)\,,\quad i=1,2\,;\;\; j=1,3\,;\;\; (m,n)=(1,1)-(m_0,n_i)\,,\;\; s=1-s^{*}\,.$$

In this case the unknowns (with indication of their number) are

$$V_{m,n}^{(i)}(t)-m_0\,(n_1+n_2)\,,\;\; f_{j,m}^{(i)}(t)-4m_0\,,\;\; f_{j+1,n}^{(i)}(t)-2\,(n_1+n_2)\,,\;\; f_{\square s}^{*}(t)-s^{*}\,,$$
$$(m,n)=(1,1)-(m_0,n_i)\,,\;\; j=1,3\,;\;\; i=1,2\,;\;\; s=1-s^{*}\,, \tag{27}$$

and 20 more unknowns

$$f_{j}^{(i)}(t,0)\,,\;\; f_{j}^{(i)}(t,a)\,,\;\; f_{j+1}^{(i)}(t,y_{0i})\,,\;\; f_{j+1}^{(i)}(t,y_{0i}+b_i)\,,\;\; f_{\square}^{*}(t,0)\,,\;\; f_{\square}^{*}(t,a)\,,\;\; \theta_u^{*}(t)\,,\;\; (i,u)=1,2\,. \tag{28}$$

The total number of the unknown functions in (27) and (28), dependent just on the variable t, is equal to $[m_0(n_1+n_2) + 4m_0 + 2(n_1+n_2) + s^* + 20]$. Two unknown $\theta_u^*(t)$ are found as two roots of Eqs. (3). The remaining 18 quantities of those indicated in (28) are related by 12 equations (22) and (24).

To close system (27) and (28) it only remains for us to write $[m_0(n_1+n_2) + 4m_0 + 2(n_1+n_2) + s^* + 6]$ equations. For this purpose we substitute the series for V_i from (26) and M_i from (25) into the differential equations (13), multiply them by $G_{m,n}^{(i)}$, and integrate over the region Ω for i = 1 and 2. As a result we will have a system of ordinary differential equations of first order in t from $m_0(n_1+n_2)$ equations for $V_{m,n}^{(i)}(t)$:

$$\iint\limits_{\Omega_{\square i}}\Big[D_i\,(V_i+M_i)-q_{\square i}\Big]\,G_{m,n}^{(i)}dxdy=0\,,\quad (m,n)=(1,1)-(m_0,n_i)\,,\quad i=1,2\,. \tag{29}$$

The initial conditions for system (29) will be obtained from the initial conditions $f_{\square 0}^{(i)}$ given in (14). For this purpose we substitute their expressions from (20), (25), and (26) instead of $T_{\square i}$, M_i, and V_i and, similarly to the actions in (29), find the sought initial conditions

$$V_{m,n}^{(i)}(0)=\frac{4}{ab_i}\iint\limits_{\Omega_{\square i}}\Big[f_{\square 0}^{(i)}(x,y)-M_i\big|_{t=0}\Big]\,G_{m,n}^{(i)}dxdy,\quad (m,n)=(1,1)-(m_0,n_i)\,,\quad i=1,2\,. \tag{30}$$

Also, system (29) contains other unknowns given in the sets (27) and (28) because of which it should be supplemented with equations obtained from the boundary conditions at Γ_1, Γ_2, and $\Gamma_{\square}^{*}$. To fulfill the boundary conditions at Γ_i, i = 1 and 2, from (18) we must know the location of two points A_u^*, u = 1 and 2, which separate Γ into the parts Γ_1 and Γ_2. Therefore, Eq. (3) for finding two parameters θ_u^* should be included into the general system for unknown quantities. We impart a more convenient form to Eq. (3), replacing $f^*(t,\, x_\Gamma(t,\, \theta_u^*))$ by $f_{\square}^*(t,\, x_\Gamma(t,\, \theta_u^*))$ in it, since these functions are coincident at two points A_u^*:

$$y_\Gamma\left(t,\theta_u^{*}\right)=f_{\square}^{*}(t,0)+\frac{x_\Gamma\,(t,\theta_u^{*})}{a}\Big[f_{\square}^{*}(t,a)-f_{\square}^{*}(t,0)\Big]+\sum_{s=1}^{s^{*}}f_{\square s}^{*}(t)\sin s\pi\,\frac{x_\Gamma\,(t,\theta_u^{*})}{a}\,. \tag{31}$$

Two parameters θ_u^* found from (31) will be arranged in the order $\theta_1^* < \theta_2^*$; then we can indicate, for Γ_1 and Γ_2, the following two ranges:

$$\theta\in\left[\theta_1^{*},\theta_2^{*}\right]\,,\quad \theta\in\left[\theta_2^{*},\theta_1^{*}+\theta_0\right)\,. \tag{32}$$

From the initial conditions given in (9), we can accurately determine where the 1st and 2nd phases are to be found. This will enable us to establish to what boundary the ranges of variation in the parameter θ from (32) belong. Let, for definiteness, the first range in (32) belong to Γ_1 and the second belong to Γ_2.

To fulfill boundary conditions (18) at Γ_1 and Γ_2 we substitute M_i from (25) and V_i from (26) into (18) where the coordinates (x_Γ, y_Γ) will be replaced by their parametric dependences (1) in accordance with the ranges (32) which will be denoted by ($x_{\Gamma i}(t, \theta)$ and $y_{\Gamma i}(t, \theta)$), $i = 1$ and 2:

$$F_i\,(t, \theta) = f_i\,(t, \theta)\,, \quad F_i\,(t, \theta) = L_i\,(M_i + V_i)\,\big|_{(x,y)=(x_{\Gamma i}, y_{\Gamma i})}\,,$$
$$f_i\,(t, \theta) = f\,(t, x_{\Gamma i}, y_{\Gamma i})\,, \quad [f_i, F_i] \in \left[C^{(1)}\,(\Omega), L_p^{\alpha}\,(\Omega)\right]. \tag{33}$$

The left- and right-hand sides of the equalities will be expanded in sines in the corresponding ranges (32), which ensures a uniform convergence of the approximate solution. Equality of the functions in (33), according to Fejer's theorem [6], yields the equality of their Fourier coefficients, i.e.,

$$\int_{\theta_1}^{\theta_2} F_1\,(t, \theta)\,\sin k\pi\,\frac{\theta - \theta_1^*}{\theta_2^* - \theta_1^*}\,d\theta = \int_{\theta_1}^{\theta_2} f_1\,(t, \theta)\,\sin k\pi\,\frac{\theta - \theta_1^*}{\theta_2^* - \theta_1^*}\,d\theta\,,$$
$$\int_{\theta_2}^{\theta_0+\theta_1^*} F_2\,(t, \theta)\,\sin k\pi\,\frac{\theta - \theta_2^*}{(\theta_0 + \theta_1^*) - \theta_2^*}\,d\theta = \int_{\theta_2}^{\theta_0+\theta_1^*} f_2\,(t, \theta)\,\sin k\pi\,\frac{\theta - \theta_2^*}{(\theta_0 + \theta_1^*) - \theta_2^*}\,d\theta\,, \tag{34}$$
$$k = 1 - \left(2m_0 + n_1 + n_2 - s^*\right).$$

When sine expansions in accordance with formulas of the (26) type are used, we should add, to system (34), four equalities resulting from (33) for a θ value equal to the values at the ends of their ranges (32):

$$F_i\left(t, \theta_i^*\right) = f_i\left(t, \theta_i^*\right), \quad F_1\left(t, \theta_2^*\right) = f_1\left(t, \theta_2^*\right), \quad F_2\left(t, \theta_2^* + \theta_0\right) = f_2\left(t, \theta_2^* + \theta_0\right), \quad i = 1, 2\,. \tag{35}$$

The entire system (34) and (35) contains $2(2m_0 + n_1 + n_2 - s^*)$ equations for the quantities of (27) and (28). It only remains for us to fulfill conditions (15) at the phase boundary $\Gamma_\square^*$, which will be rewritten in a more convenient form:

$$\Phi_i\,(t, x) = \varphi_i\,(t, x)\,, \quad \Phi_i\,(t, x) = T_{\square i}\,\big|_{\Gamma_\square^*}\,, \quad \varphi_i\,(t, x) = T^*\,,$$
$$\Phi_3\,(t, x) = \varphi_3\,(t, x)\,, \quad \varphi_3\,(t, x) = h^*\,\frac{\partial f_\square^*}{\partial t}\bigg|_{\Gamma_\square^*}\,, \quad i = 1, 2\,; \tag{36}$$
$$\Phi_3\,(t, x) = \left[\lambda_1\left(\frac{\partial T_1}{\partial y} - \frac{\partial T_1}{\partial x}\,\frac{\partial f_\square^*}{\partial t}\right) - \lambda_2\left(\frac{\partial T_2}{\partial y} - \frac{\partial T_2}{\partial x}\,\frac{\partial f_\square^*}{\partial t}\right)\right]\Bigg|_{\Gamma_\square^*}.$$

We expand the left- and right-hand sides of three equalities (36) in sines in the range $[0, a]$. From the equality of their Fourier coefficients, we will have

$$\int_0^a \Phi_j\,(t, x)\,\sin s\pi\,\frac{x}{a}\,dx = \int_0^a \varphi_j\,(t, x)\,\sin s\pi\,\frac{x}{a}\,dx\,, \quad j = 1 - 3\,, \quad s = 1 - s^*\,. \tag{37}$$

To system (37), we should add six more equalities obtained from (36) in computing the values of the functions at the ends of the range:

$$\Phi_j(t, 0) = \varphi_j(t, 0), \quad \Phi_j(t, a) = \varphi_j(t, a), \quad j = 1 - 3. \tag{38}$$

The entire system (37) and (38) contains $3(s^* + 2)$ equations. The last $s^* + 2$ equations in (37) and (38) for $j = 3$ are ordinary differential equations of first order for $f^*_\square(t, 0)$, $f^*_\square(t, a)$, and $f^*_{\square m}(t)$. We find the initial conditions for them from the corresponding initial conditions given in (2), which will be rewritten, using the supplemented definition (17), in the form

$$f^*_{\square 0}(x) = f^*_\square(0, 0) + \frac{x}{a}\left[f^*_\star(0, a) - f^*_\square(0, 0)\right] + \sum_{s=1}^{s^*} f^*_{\square s}(0) \sin s\pi \frac{x}{a}. \tag{39}$$

Setting $x = 0$ and $x = a$ in (39), we have two equations; thereafter, multiplying (39) by $\sin s\pi x/a$ and integrating over the region $0 \le x \le a$, we obtain additionally s^* equations, a total of $s^* + 2$ equalities as the sought initial conditions for system (37) and (38):

$$f^*_\square(0, 0) = f^*_{\square 0}(0), \quad f^*_\square(0, a) = f^*_{\square 0}(a), \quad s = 1 - s^*; \tag{40}$$

$$f^*_{\square s}(0) = \frac{2}{a}\int_0^a \left\{f^*_{\square 0}(x) - f^*_{\square 0}(0) - \frac{x}{a}\left[f^*_{\square 0}(a) - f^*_{\square 0}(0)\right]\right\} \sin s\pi \frac{x}{a}\, dx.$$

Conclusions. To find $[m_0(n_1 + n_2) + 4m_0 + 2(n_1 + n_2) + s^* + 20]$ unknowns we have the same number of equations: Eq. (31) for computation of two parameters θ^*_u, $m_0(n_1 + n_2)$ equations with initial conditions (30) for determination of $V^{(i)}_{m,n}$ in (29), $3(s^* + 2)$ equations in (37) and (38), and $2(2m_0 + n_1 + n_2 - 2s^*)$ more equations and 12 equations from (22) and (24) in (34). The above system enables us to obtain the approximate solution of the phase-transformation problem in analytical form. The improved Fourier series used uniformly and quite rapidly converge. Thus, e.g., if the functions have a curvature of constant sign, no more than ten terms can safely be allowed for in these series.

Algorithm for the Employment of the Method

1. Write the equation of the boundary Γ in parametric form (1).
2. Reduce the formulation of the phase-transformation problem to the form (8)–(10).
3. Determine the smallest dimensions of the rectangles $\Omega_{\square 1}$ and $\Omega_{\square 2}$, which must contain the regions Ω_1 and Ω_2 at $t \in [0, t_0]$.
4. The rectangles $\Omega_{\square 1}$ and $\Omega_{\square 2}$ must have the same dimension along the x axis; the possibility of their partial overlapping is allowed.
5. Supplement the definition of the initial conditions $f^{(i)}_{\square 0}$ and $f^*_{\square 0}$ of the right-hand side of the differential equation (14) $q_{\square i}$ in the region $\Omega_{\square i} \setminus \Omega$.
6. Write the boundary condition of the auxiliary problem for the region $\Omega_{\square i}$ in general form (21) in terms of the functions $f^{(i)}_j$.
7. Represent the boundary functions M_i in the form (25).
8. Write matching conditions (24).
9. Represent the unknown functions V_i, $f^{(i)}_j$, $f^{(i)}_{j+1}$, and $f^*_\square$ in the form of improved Fourier series (26) with a limited number of terms.
10. Compose a closed system of Eqs. (22), (24), (29), (31), (34), (37), and (38) for two unknown θ^*_u and the coefficients indicated in (27) and (28).
11. In solving the above system, it is necessary to use initial conditions (30) and (40).
12. Substitute the found quantities into expressions (26) for V_i and (25) for M_i on solution of this system.
13. Substitute the found quantities M_i and V_i into (20) and obtain the solution of the problem for $T_{\square i}$.
14. Determine the position of the phase-transformation boundary Γ^* from the equation $y = f^*_\square(t, x)$ where $f^*_\square(t, x)$ is taken from (26).

In considering nonlinear boundary-value problems for curvilinear regions without phase transition, one should omit all actions related to the moving phase boundary in the algorithm. The method of extension of boundaries is substantially simplified.

NOTATION

A_u^*, two points at the intersection of three boundaries Γ_1, Γ_2, and Γ^*; (a, b_i), dimensions of the rectangles $\Omega_{\square i}$, m; $\tilde{a}$, thermal diffusivity, m^2/sec; $B_{\square u}^*$, two points at the intersection of three boundaries $\Gamma_{\square 1}$, $\Gamma_{\square 2}$, and $\Gamma_{\square 1}^*$; $C^{(1)}$ and $C^{(2)}$, spaces of differentiable functions; D_i, nonlinear differentiable operators of 2nd order in the coordinates x and of 1st order in the time t; $f_{\square \Gamma i}$, boundary conditions in general form at $\Gamma_{\square i}$; $f_0^{(i)}$, initial conditions of the initial problem for the regions Ω_i; $f_{\square 0}^{(i)}$, initial conditions for the extended regions $\Omega_{\square i}$; $f_{\square 0}^*$, initial position of the phase-transformation boundary in the regions $\Omega_{\square i}$; $f_{\square}^*$, equation of the phase-transformation boundary in the extended regions $\Omega_{\square i}$; $f_j^{(i)}$ (j = 1–4 and i = 1 and 2), unknown boundary conditions on the sides of the rectangles $\Gamma_{\square i}$; f^*, equation of the phase-transformation boundary for the initial problem; f_0^*, initial position of the phase-transformation boundary in the initial problem; $f_{\Gamma i}$, boundary conditions in general form at the boundaries Γ_i of the regions Ω_i; $f_i(t, \theta)$ and $F_i(t, \theta)$, auxiliary functions used when the boundary conditions at Γ_i are fulfilled; $f_{j,m}^{(i)}$, $f_{\square m}^*$, $q_{m,n}^{(i)}$, and $V_{m,n}^{(i)}$, Fourier coefficients of the functions $f_j^{(i)}$, $f_{\square}^*$, q_i, and V_i in the corresponding domains of their definition; $g_*(t, y)$, equation of the phase-transformation boundary, expressed relative to the variable x; h^*, specific heat of phase transformation, J/kg; L_p^{α}, classes of Sobolev–Liouville functions; M_i, boundary functions; m_0, n_1, n_2, and s^*, number of retained terms in Fourier series; N_0, auxiliary quantity used in determining the velocity of motion of the phase boundary; n^*, vector of unit normal to $\Gamma_{\square m}^*$; q_i and $q_{\square i}$, known right-hand sides of the differential equations (8) and (14), having the meaning of sources; $T_{\square i}$, temperatures inside the regions $\Omega_{\square i}$; t, time, sec; t_0, time of consideration of the process, sec; v_n, normal component of the velocity of motion of the boundary Γ^*, m/sec; V_i, functions satisfying the homogeneous boundary conditions at $\Gamma_{\square i}$; $V_{m,n}^{(i)}$, $f_{j,m}^{(i)}$, and $f_{\square s}^*$, coefficients of spectral Fourier expansions; x_{Γ}, coordinates of points of the boundary Γ or Γ_i in the corresponding problem; $(x_{\Gamma \square i}, y_{\Gamma \square i})$, moving boundary of the region $\Gamma_{\square i}$; (x_{au}^*, y_{au}^*), coordinates of two points A_u^* at the intersection of three boundaries Γ_1, Γ_2, and Γ^*; (x_{bu}^*, y_{bu}^*), coordinates of two points B_u^* at the intersection of three boundaries $\Gamma_{\square 1}$, $\Gamma_{\square 2}$, and $\Gamma_{\square}^*$; x, y, Cartesian coordinates; Γ, moving boundary of the region Ω; Γ_i, boundary of the phase region Ω_i; $\Gamma_{\square i}$, boundary of the phase region $\Omega_{\square i}$; $\Gamma_{\square}^*$, phase-transformation boundary in $\Omega_{\square}$; θ, parameter in the equation of the boundary Γ; θ_0, range of variation in the parameter θ; θ_u^*, values of the parameter θ at the points A_u^* (u = 1 and 2); λ_1 and λ_2, thermal conductivities, W/(m·K); $\Phi_j(t, x)$ and $\varphi_j(t, x)$, auxiliary functions used when the boundary conditions at the phase boundary $\Gamma_{\square}^*$ are fulfilled; Ω, arbitrary curvilinear region. Subscripts: i, number of the phase region Ω_i, i = 1 and 2; j, number of the side of a rectangle; m and n, numbers of the ranges of eigenfunctions and eigenvalues for a rectangle; s, summation index; u, number for quantities that refer to the points of intersection of the boundary $\Gamma_{\square}^*$ and Γ_i or $\Gamma_{\square i}$; $\square$, quantities that refer to the auxiliary problem in the extended region $\Omega_{\square}$; 0, initial conditions or fixed quantities t_0, θ_0, m_0, and n_0. Superscripts: (i), quantities that refer to the phase regions Ω_i, i = 1 and 2; *, quantities that refer to the phase-transformation boundary Γ^*.

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